Entropy methods for kinetic models of traffic flow

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Abstract

In these notes we first introduce logarithmic entropy methods for time-dependent drift-diffusion equations and then consider a kinetic model of Vlasov-Fokker-Planck type for traffic flows. In the spatially homogeneous case the model reduces to a special type of nonlinear drift-diffusion equation which may permit the existence of several stationary states corresponding to the same density. Then we define general convex entropies and prove a convergence result for large times to steady states, even if more than one exist in the considered range of parameters, provided that some entropy estimates are uniformly bounded.

Keywords. Traffic flow – time-dependent diffusions – drift-diffusion equations – nonlinear friction and diffusion coefficients – entropy method – relative entropy – large time asymptotics

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Introduction

First order models are well suited to describe certain regimes of traffic flows. Compared to these models, the interest of kinetic approaches is that they allow a better description of the behaviour of the drivers at the individual level [3, 5]. It has recently been shown in [3] that a rather simple Vlasov-Fokker-Planck model involving a nonlinear diffusion can explain the existence of multiple steady solutions.

Over the last five years there has been much interest in entropy methods for parabolic equations [1]. Although some of these techniques were well known in probability theory, it is surprising to see the variety of models which can be tackled with these methods. The main reason for this success in both linear and nonlinear settings is likely resident in the gradient flow properties [6] which are natural for parabolic equations and translate, for example, readily into a general contraction property [2].

The goal of these notes is to prove, by entropy methods, that the solution of a nonlinear drift-diffusion equation, with generally more than one stationary solution, converges for large times to a single steady state. This applies in particular to the homogeneous version of the novel traffic dynamics models mentioned above.

This paper starts with considerations on logarithmic entropies for linear diffusions with time-dependent coefficients. The Fokker-Planck model, which is the homogeneous version of the Vlasov-Fokker-Planck model of traffic dynamics introduced in [3, 4] is then described and the convergence to a stationary solution is established in the so-called Maxwellian case. A more general family of convex relative entropies, with a time-dependent normalization coefficient, is then introduced in Section 3 and allows us to describe the two possible asymptotic regimes for large times: convergence to a stationary solution or large entropy regime. Further considerations on concentration *vs* convergence are given in the last part of this paper.

1 Linear diffusions and a logarithmic entropy

Consider a solution $f = f(t, v)$ of the linear diffusion equation

$$f_t = (-B(t, v)\, f + D(t, v)\, f')' \;, \quad (t, v) \in \mathbb{R}^+ \times (0, 1) \tag{1}$$

where f_t and f' respectively denote the derivatives of f with respect to t and v. We shall assume that D is nonnegative and such that

$$C(t, v) = -\int_0^v \frac{B(t, w)}{D(t, w)}\, dw \tag{2}$$

is well defined for any $(t, v) \in \mathbb{R}^+ \times (0, 1)$, and that $(t, v) \mapsto e^{-C(t,v)}$ belongs to $L^\infty(\mathbb{R}^+, L^1(0, 1))$. Let f_0 be a nonnegative initial datum for (1) and define

$$\rho = \int_0^1 f_0(v)\, dv \;.$$

Denote by F_f the flux associated to f:

$$F_f(t, v) := -B(t, v)\, f(t, v) + D(t, v)\, f'(t, v) \;.$$

Under the no-flux boundary conditions

$$F_f(t, 0) = F_f(t, 1) = 0 \quad \forall\, t \geq 0 \;,$$

the total mass is preserved:

$$\int_0^1 f(t, v)\, dv = \rho \quad \forall\, t \geq 0 \;.$$

This allows us to define a local equilibrium

$$g(t, v) = \rho\, \frac{e^{-C(t,v)}}{\int_0^1 e^{-C(t,w)}\, dw} \tag{3}$$

which has zero flux:

$$F_g(t, v) = 0 \quad \forall\, (t, v) \in \mathbb{R}^+ \times (0, 1) \;,$$

but which is not in general a stationary solution of (1), since $g_t \equiv 0$ is not granted. We may also replace $C(t, v)$ by $C(t, v) + C_0(t)$ for any function $C_0(t)$ without changing the values of $g(t, v)$: the choice of the primitive of $-B/D$ in (2) is free. Note that g is the unique minimizer of the functional

$$f \mapsto \int_0^1 f(\log f + C(t, v))\, dv$$

on $L^1_+(0,1)$ under the constraint $\int_0^1 f\, dv = \rho$. At this point, we can define a notion of relative entropy as follows. Let

$$e[t,f] := \int_0^1 (f \log f - g \log g + C(t,v)(f-g))\, dv - \iint_{(0,1)\times(0,t)} C_t(s,v)(f-g)(s,v)\, dv\, ds\ .$$

A straightforward computation shows that if f is a solution of (1) and if all above conditions are fulfilled, then

$$\frac{d}{dt} e[t, f(t,.)] = -\int_0^1 D(t,v)\, f \left| \frac{f'}{f} - \frac{g'}{g} \right|^2 dv\ .$$

This computation is of course formal and yields no conclusions, since we have in general no control on any lower bound of $e[t, f(t,.)]$, especially when we shall generalize (1) to a nonlinear model in which the coefficients B and D depend on f through the mean velocity.

2 The homogeneous Fokker-Planck model of traffic dynamics

The homogeneous Fokker-Planck model of traffic dynamics corresponds to Eq. (1) in the case $\rho \in (0,1]$ and where B and D are defined in a nonlinear way as functions of the average velocity

$$u(t) = \frac{1}{\rho} \int_0^1 v\, f(t,v)\, dv \tag{4}$$

as follows:

$$B(t,v) = \begin{cases} -\, C_B\, |v-u(t)|^2 \rho \left(1 - \left| \frac{v-u(t)}{1-u(t)} \right|^\delta \right) & \text{if } v > u(t) \\ C_A\, |v-u(t)|^2 (1-\rho) & \text{if } v \le u(t) \end{cases} \tag{5}$$

and

$$D(t,v) = \sigma\, m_1(\rho)\, m_2(u(t))\, |v-u(t)|^\gamma \tag{6}$$

for some positive constants C_A, C_B, σ and nonnegative exponents γ and δ. In [3] m_2 was chosen as $m_2(u) = u(1-u)$, and m_1 a function which consists of two linked pieces of Gaussian distributions such that $\max_{(0,1)} m_1 = m_1(0.3)$ and $m_1(0) = m_1(1) = 0$. For these choices the following result has been obtained in [3]. We also refer to [4] for further comments on the model.

Proposition 1 [3] *Any stationary solution is uniquely determined by ρ and its average velocity u. The set of stationary solutions is therefore represented by a diagram $(\rho, u[\rho])$, which is in general multivalued. For any $\rho \in (0,1]$, there exists at least one stationary solution. For some parameter choices (for example $\gamma = \delta = 1$; see [3]), there exist ρ_1, $\rho_2 \in (0,1)$ such that there is exactly one solution for $\rho \in (0,1) \setminus (\rho_1, \rho_2)$ and three solutions for $\rho \in (\rho_1, \rho_2)$.*

Example. *The Maxwellian case.* This is a special case which can be handled with the technique of Section 1. Assume that u, B and D are defined by (4), (5) and (6) with $\delta = 0$, $C_B \rho = C_A(1-\rho) := K$, $m_2 \equiv 1$ and $\gamma = 1$. Then $B = K\, |u-v|(u-v)$,

$$C = \frac{K}{2\sigma m_1(\rho)}\, |u-v|^2 + C_0(t)$$

and

$$\int_0^1 C_t(t,v)(f-g)\,dv = (C_0)_t \int_0^1 (f-g)\,dv + \frac{K}{\sigma m_1(\rho)}\,\frac{du}{dt}\cdot \int_0^1 (v-u(t))\,(f-g)\,dv\;.$$

On the one hand, $\int_0^1 (v-u(t))f\;dv = 0$ by definition of $u(t)$, and on the other hand there exists a function F such that

$$\frac{d}{dt}F(u(t)) = F'(u(t))\,\frac{du}{dt} = \frac{K}{\sigma m_1(\rho)}\,\frac{du}{dt}\,\int_0^1 (v-u(t))\,g\;dv\;,$$

since g itself is a function of $v-u(t)$, namely $F(u) = \log(\int_0^1 \exp\{-\frac{K}{\sigma m_1(\rho)}\,|v-u(t)|^2\}\,dv)$. We can therefore write

$$e[t,f] := \int_0^1 \left(f\log f - g\log g + C(t,v)(f-g)\right)\;dv - F(u(t))\;.$$

F is bounded on $[0,1]\ni u(t)$. By using a reasoning as in Section 4 below one can prove that $f(t,\cdot)$ converges as $t\to+\infty$ to the (unique) stationary solution of (1) which takes the form (3) for some unique $u = u[\rho]$. Here are the main steps of the proof. Since $\log g = -C + Constant$,

$$\begin{aligned}\int_0^1 \Big(f\log f - g\log g + C(t,v)(f-g)\Big)\;dv &= \int_0^1 f\log\left(\frac{f}{g}\right)\;dv\\ &\geq \int_0^1 f\;dv\;\log\left(\frac{\int_0^1 f\,dv}{\int_0^1 g\,dv}\right) = 0\end{aligned}$$

by Jensen's inequality: $e[t,f]$ is therefore bounded from below and converges to some limit as $t\to+\infty$, which means that $\int_0^1 D(t,v)\,f\,\left|f'/f - g'/g\right|^2\;dv$ converges to zero. It is then easy to conclude that $f = g$ is a stationary solution. The proof of the uniqueness of the stationary solution is left to the reader.

This approach generalizes to any equation of type (1) such that

$$\frac{B}{D} = -k\,(v-u(t))\;,$$

for some positive constant k, whatever D is. We have of course to assume that $\gamma < 3$, so that C is well defined by (2) and uniformly bounded w.r.t. $(t,v)\in \mathbb{R}_+\times(0,1)$. The specific form of D then plays a crucial role if one wants to determine the rate of convergence to the stationary solution.

3 Convex entropies

We are going to consider more general entropies than in Section 1, which apply to (1) both in the linear and in the nonlinear case described in Sections 1 and 2 respectively.

Define the *relative entropy of f with respect to g* by

$$E[\,f\,|\,g\,] = \int_0^1 \Phi\Big(\frac{f}{g}\Big)\,g\;dv\;, \tag{7}$$

for some convex function Φ. Let for instance $\Phi(u) = u^\alpha$ for some $\alpha > 1$. This is our "standard" example. We recover logarithmic entropies by taking $\Phi(u) = u\log u$ (this corresponds to a limit case that we shall denote by "$\alpha = 1$"). Both examples satisfy in addition $\Phi(0) = 0$.

Taking into account the boundary conditions, we can rewrite (1) as follows:

$$\begin{cases} f_t = \left[D(t,v)\, f \left(\frac{f'}{f} - \frac{g'}{g} \right) \right]' = \left[D(t,v)\, g \left(\frac{f}{g} \right)' \right]' & \forall\, (t,v) \in \mathbb{R}^+ \times (0,1) \\ \left(\frac{f}{g} \right)' (t,v) = 0 & \forall\, t \in \mathbb{R}^+,\ v = 0,\ 1 \end{cases} \tag{8}$$

Here $g = g(t,v)$ is given by

$$g(t,v) = \kappa(t)\, e^{-C(t,v)} \tag{9}$$

for some $\kappa(t) \neq 0$ to be fixed later (and not given by (3) any more). With f and g given by (8) and (9) respectively,

$$\frac{d}{dt} E[\, f(t,\cdot) \,|\, g(t,\cdot)\,] = \int_0^1 \Phi'\Big(\frac{f}{g}\Big) f_t \; dv + \int_0^1 \left[\Phi\Big(\frac{f}{g}\Big) - \frac{f}{g}\, \Phi'\Big(\frac{f}{g}\Big) \right] g_t \; dv \; .$$

Using (8), we can evaluate the first term of the r.h.s. by

$$\int_0^1 \Phi'\Big(\frac{f}{g}\Big) f_t \; dv = - \int_0^1 \Phi''\Big(\frac{f}{g}\Big)\, D(v,t)\, g\, \left| \Big(\frac{f}{g}\Big)' \right|^2 \; dv \; .$$

Let $\Psi(u) = \Phi(u) - u\Phi'(u) < 0$, for any $u > 0$ (this is true for our standard examples and for any strictly convex Φ such that $\Phi(0) = 0$). The second term is

$$\begin{aligned} \int_0^1 \left[\Phi\Big(\frac{f}{g}\Big) - \frac{f}{g}\, \Phi'\Big(\frac{f}{g}\Big) \right] g_t \; dv \;\; &= \;\; \int_0^1 \Psi\Big(\frac{f}{g}\Big)\, g\, \frac{g_t}{g} \; dv \\ &= \;\; \int_0^1 \Psi\Big(\frac{f}{g}\Big)\, g \left(\frac{\dot\kappa}{\kappa} - C_t(t,v) \right) \; dv \end{aligned}$$

where we used the notation $\dot\kappa = \frac{d\kappa}{dt}$. The right hand side vanishes if we choose

$$\begin{cases} \dot\kappa = \kappa\, \dfrac{\int_0^1 \Psi(\frac{f}{g})\, g\, C_t(t,v)\; dv}{\int_0^1 \Psi(\frac{f}{g})\, g \; dv} \\ \kappa(0) = 1 \; . \end{cases} \tag{10}$$

Note that C depends on u and therefore on f, even in the standard cases $\Phi(u) = u^\alpha$ or $u \log u$ ($\alpha = 1$). In the limit case $\alpha = 1$, g is not directly involved in the definition of κ:

$$\dot\kappa = \frac{\kappa}{\rho} \int_0^1 f(t,v)\, C_t(t,v) \; dv \; ,$$

since $\Psi(u) = -u$ is linear.

Lemma 2 *Consider a solution of (1). Assume that f and C are smooth and globally defined in t. If Φ is convex, nonnegative, superlinear at $+\infty$ and such that $\Psi(u) = \Phi(u) - u\Phi'(u)$ has no zero, then κ is bounded away from zero. If moreover C_t is, locally in t, bounded, then (10) has a global positive solution.*

Proof. As long as κ is positive and well defined,

$$E[\, f(t,\cdot) \,|\, g(t,\cdot)\,] \leq E[\, f(0,\cdot) \,|\, g(0,\cdot)\,]$$

as we shall see below, so that by Jensen's inequality

$$\kappa(t) \int_0^1 e^{-C} \, dv \;\; \Phi \left(\frac{\rho}{\kappa(t)\, \int_0^1 e^{-C}\, dv} \right) \leq E[\, f(0,\cdot) \,|\, g(0,\cdot)\,] \; .$$

This is possible if and only if $\kappa(t)$ is bounded away from zero.

On the other hand, with C_t locally bounded in t,

$$\frac{d}{dt}\Big(\log\kappa\Big)\leq\|C_t(t,\cdot)\|_{L^\infty(0,1)}\;,$$

which proves that κ is locally bounded in t. □

For the choice of κ corresponding to (10), $E[\,f(t,\cdot)\,|\,g(t,\cdot)\,]$ is nonnegative (by construction) if Φ is nonnegative, and nonincreasing :

$$\frac{d}{dt}E[\,f(t,\cdot)\,|\,g(t,\cdot)\,]=-\int_0^1\Phi''\Big(\frac{f}{g}\Big)\,D(v,t)\,g\,\left|\Big(\frac{f}{g}\Big)'\right|^2\,dv=:-I[\,f(t,\cdot)\,|\,g(t,\cdot)\,]\;.$$

For the convenience of the reader, let us give the explicit form of the entropy and the entropy production terms in the standard cases.

Lemma 3 *Let f be a smooth function on $(0,1)$.*
(i) *if $\Phi(u)=u^\alpha$ for some $\alpha>1$, then*

$$E[\,f\,|\,g\,]=\int_0^1 f^\alpha g^{1-\alpha}\,dv\;,$$

(ii) *if $\Phi(u)=u\log u$, then*

$$E[\,f\,|\,g\,]=\int_0^1 f\log\Big(\frac{f}{g}\Big)\,dv\;,$$

and in both cases, with $\epsilon(\alpha)=\alpha(\alpha-1)$ if $\alpha>1$ and $\epsilon(\alpha)=1$ if $\alpha=1$,

$$\begin{aligned}I[\,f\,|\,g\,]&=\epsilon(\alpha)\int_0^1 D\,f^{\alpha-2}g^{3-\alpha}\,\left|\Big(\frac{f}{g}\Big)'\right|^2dv\\&=\epsilon(\alpha)\int_0^1 D\,f^\alpha g^{1-\alpha}\,\left|\frac{f'}{f}-\frac{g'}{g}\right|^2dv\\&=4\,\frac{\epsilon(\alpha)}{\alpha^2}\int_0^1 D\,g^{1-\alpha}\,\left|(f^{\alpha/2})'+\frac{\alpha}{2}\,C'f^{\alpha/2}\right|^2dv\;.\end{aligned}$$

If Φ changes sign but is bounded from below and superlinear (for example, if $\alpha=1$: see Remark 5), the boundedness from below is a consequence of Jensen's inequality. This proves the following

Proposition 4 *Assume that Φ is convex, bounded from below, superlinear at ∞ and that $\Psi(u):=\Phi(u)-u\Phi'(u)$ has no zero. Assume that C_t is bounded in $L^\infty((0,T)\times(0,1))$ for any $T>0$ and consider a smooth solution f of (8), with E, g, and κ defined by (7), (9) and (10). Then*

$$E[\,f(t,\cdot)\,|\,g(t,\cdot)\,]+\int_0^s\int_0^1\Phi''\Big(\frac{f}{g}\Big)\,D(v,s)\,g\,\left|\Big(\frac{f}{g}\Big)'\right|^2\,dv\,ds=E[\,f(0,\cdot)\,|\,g(0,\cdot)\,]\;.$$

As a consequence, with $E[\,f(0,\cdot)\,|\,g(0,\cdot)\,]=:\mathcal{E}_0$,
(i) *if $\Phi(u)=u^\alpha$ for some $\alpha>1$, then*

$$4\,\frac{\alpha-1}{\alpha}\int_0^t\!\!\int_0^1 D\,g^{1-\alpha}\,\left|(f^{\alpha/2})'+\frac{\alpha}{2}\,C'f^{\alpha/2}\right|^2dv\,ds+\big(\kappa(t)\big)^{1-\alpha}\int_0^1 f^{\alpha-1}e^{(\alpha-1)C}dv=\mathcal{E}_0\;,$$

(ii) *if $\Phi(u)=u\log u$ (case $\alpha=1$), then*

$$\int_0^t\int_0^1 D\,f\,\left|\frac{f'}{f}-\frac{g'}{g}\right|^2\,dv\,ds+\int_0^1 f(t,v)(\log f(t,v)+C(t,v))\,dv-\rho\log\kappa(t)=\mathcal{E}_0\;.$$

The results of Lemma 2 and Proposition 4 apply to the nonlinear model of Section 2. One has to translate the assumptions made on C into assumptions on m_2 and u. This is left to the reader. The main difficulty to obtain informations on the large time behaviour is to replace local estimates by global ones (see Remark 7).

Remark 5 *Using the fact that $u \mapsto u^\alpha - 1 - \alpha(u-1) =: \varphi(u)$ if $\alpha > 1$, $u \mapsto u \log u - u + 1 =: \varphi(u)$ if $\alpha = 1$, reaches its minimum at $u = 1$, it is tempting to use a refined version of the entropy based on φ instead of the one based on Φ. However, this leads to problems with the choice of κ. In any of these two cases cases,*

$$E[\, f \,|\, g\,] \geq \int_0^1 g\; dv + \alpha \int_0^1 (f-g)\; dv$$

with equality if and only if $f = g$ a.e. This is not very useful if $\alpha > 1$ because we already know that $E[\, f \,|\, g\,] \geq 0$, by definition of E, but this provides an estimate if $\alpha = 1$.

As noted above, other entropies could be considered: the basic requirements of our method are that Φ is convex, bounded from below, that $\Psi(u) = \Phi(u) - u\Phi'(u)$ has a definite sign. One also needs that Φ is superlinear at infinity (see Section 4) to get convergence results. To get sharp bounds, one can of course ask that $\Phi(1) = \Phi'(1) = 0$, or replace Φ by $u \mapsto \Phi(u) - \Phi(1) - \Phi'(1)(u-1)$, but the price to pay for this modification is, as mentioned above, the difficulty of finding a solution of the ODE for κ.

Symmetrized entropies like

$$\int_0^1 \left(\Phi\Big(\frac{f}{g}\Big)\, g + \Phi\Big(\frac{g}{f}\Big)\, f \right) dv$$

could also play an important role but it is likely that the normalization condition imposed on g, i.e., the definition of κ, breaks the symmetry and shows the equivalence of the symmetrized version with the asymmetric one. This is at least the case if one considers $\Phi(u) = u \log u + 1 - u$.

4 Large time behaviour

Assuming that there is a global smooth solution of (1) in the nonlinear case where u, B and D are given by (4), (5) and (6), there are two possible asymptotic regimes depending whether κ is bounded or not as $t \to +\infty$.

4.1 Convergence to a stationary solution

Assume first that

$$\limsup_{t\to+\infty} \kappa(t) < +\infty\ .$$

The next theorem is our main result.

Theorem 6 *Assume that Φ is convex, bounded from below, superlinear at infinity and that $\Psi(u) := \Phi(u) - u\Phi'(u)$ has no zero. Consider a smooth global in time solution f of (1) with u, B and D depending nonlinearly on f according to (4), (5) and (6), for some smooth function m_2. Let $\rho = \int_0^1 f(t,v)\, dv \in (0,1]$. Assume that $E[\, f \,|\, g\,]$ is well defined and C^1 in t, with E, g, and κ defined by (7), (9) and (10). Assume furthermore that $\gamma < 3$ and that $\limsup_{t\to\infty} \kappa(t) < \infty$. Then, as $t \to +\infty$, $f(t,\cdot)$ converges a.e. to a stationary solution f_∞ of (1) where $\int_0^1 f_\infty\, dv = \rho$, $u_\infty := \frac{1}{\rho}\int_0^1 v\, f_\infty(t,v)\; dv = u[\rho]$, with the notations of Proposition 1.*

Proof. For simplicity, let us only consider the case $\Phi(u) = u^\alpha$ with $\alpha > 1$. For κ as introduced by (10) the relative entropy is

$$E[f|g] = \kappa(t)^{1-\alpha} \int_0^1 f^\alpha e^{(\alpha-1)C(t,v)}\; dv\ ,$$

with C, B, D given by (2), (5), and (6). As $\gamma < 3$, there is a constant $R > 0$ such that for any $(t,v) \in \mathbb{R}^+ \times (0,1)$, $C(t,v) \in [-R, R]$. This entails positive upper and lower bounds on $e^{(\alpha-1)C(t,v)}$.

Because of Lemma 2, there is a $\kappa_0 > 0$ such that $\inf_{t>0} \kappa(t) > \kappa_0$.

Remark 7 *Unfortunately we have no information whether or not $\kappa(t)$ will always stay bounded from above; hence the extra assumption that $\limsup_{t\to\infty} \kappa(t) < \infty$. The asymptotic behaviour of κ is related to the behaviour of C_t, over which we have not been able to obtain any control. If $\limsup_{t\to\infty} \kappa(t) = \infty$ then the decrease of $E[f|g]$ is consistent with the formation of concentrations in $f(t,\cdot)$ (see the next subsection). Whether such concentrations actually arise is a question which we cannot answer at the present time. The assumptions made in the theorem exclude them.*

In the rest of the proof of Theorem 6, we only briefly sketch the main steps. Some technical details are left to the reader.

Consider an increasing unbounded sequence $(t_n)_{n\in\mathbb{N}}$. At each step, one may extract subsequences, that we shall for simplicity again denote by $(t_n)_{n\in\mathbb{N}}$. Let

$$f_n(t,v) = f(t+t_n, v) \quad \forall\, (t,v) \in (0,1)^2 \ .$$

Since $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty((0,1;dt), L^1 \cap L^\alpha(0,1;dv))$ and since $(E[\,f_n\,|\,g_n\,])_{n\in\mathbb{N}}$, with $g_n = g(\cdot + t_n, \cdot)$, is also bounded in $L^\infty((0,1;dt), L^1(0,1;dv))$, then $(f_n)_{n\in\mathbb{N}}$ is weakly relatively compact in $L^\alpha((0,1)^2)$. Denote by f_∞, the weak limit of $(f_n)_{n\in\mathbb{N}}$, up to the extraction of a subsequence. Notice that f_∞ is possibly t-dependent.

Let $u_n := \frac{1}{\rho}\int_0^1 v\, f_n\, dv$. At least * weakly in $L^\infty(0,1;dt)$, u_n converges to $u_\infty := \frac{1}{\rho}\int_0^1 f_\infty\, dv$. According to Lemma 2 and Proposition 4 (i),

$$\lim_{n\to\infty} \int_0^1 \int_0^1 D(v, t+t_n)\, g_n \left| \left[\left(\frac{f_n}{g_n}\right)^{\alpha/2} \right]' \right|^2 dv\, dt = 0 \ .$$

This means that $(f_n^{\alpha/2})_{n\in\mathbb{N}}$ actually weakly converges in $L^\infty((0,1;dt), H^1(0,1;dv))$. To be rigourous, one has to take into account a weight $|v - u_n(t)|^\gamma$, which does not play any part because of the convergence of u_n to u_∞ (see below). Again, up to the extraction of a subsequence, we may assume that $(f_n^{\alpha/2})_{n\in\mathbb{N}}$ strongly converges in $L^2((0,1)^2)$, and a.e., to f_∞. If we multiply (1) by v and integrate with respect to v, we get

$$\rho\, \frac{du_n}{dt} = \int_0^1 v\, f_n(t,v)\, dv = \int_0^1 B(t,v)\, f(t,v)\, dv - \int_0^1 D(t,v)\, f'(t,v)\, dv \ ,$$

which is clearly bounded in $L^\infty(0,1;dt)$. Applying Arzela-Ascoli's lemma, we obtain the uniform convergence of u_n to u_∞, up to the extraction of a subsequence.

Using again Proposition 4 (i), we get

$$\lim_{n\to\infty} \int_0^1 \int_0^1 D(v, t+t_n)\, g_n^{1-\alpha} \left| \Big((f_n)^{\alpha/2} \Big)' + \frac{\alpha}{2}\, C'(t+t_n, v)\, (f_n)^{\alpha/2} \right|^2 dv\, dt = 0 \ ,$$

where $C'(t+t_n, v)$ now strongly converges to a function $C'_\infty(t,v)$ which depends on t through u_∞. Passing to the limit, we obtain

$$\Big((f_\infty)^{\alpha/2} \Big)' + \frac{\alpha}{2}\, C'_\infty(t,v)\, (f_\infty)^{\alpha/2} = 0 \tag{11}$$

so that at least in the sense of distributions, one has:

$$f_\infty(t,v) = \rho\, \frac{e^{-C_\infty(t,v)}}{\int_0^1 e^{-C_\infty(t,w)}\, dw} \quad \forall\, (t,v) \in (0,1)^2 \text{ a.e.}$$

To complete the proof, it remains to check that such a steady solution does not depend on t and is unique, (*i.e.*, independent of the initial sequence $(t_n)_{n\in\mathbb{N}}$). Because of the uniform convergence of u_n to u_∞, f_∞ is a solution of (1). With the notations of Section 1 and because of (11),

$$\partial_t f_\infty = (F_{f_\infty})' = 0 \ .$$

The uniqueness is once more a consequence of Proposition 4. Assume that we can find two sequences $(t_n^1)_{n\in\mathbb{N}}$ and $(t_n^2)_{n\in\mathbb{N}}$ with $t_n^1 < t_n^2$ for which the limiting distribution functions are two functions f_∞^1 and f_∞^2. Up to extraction of subsequences, we would find

$$\lim_{n\to\infty} \int_{t_n^1}^{t_n^2+1} \int_0^1 D(v,t)\, g^{1-\alpha} \left| \left(f^{\alpha/2}\right)' + \frac{\alpha}{2}\, C'(t,v)\, f^{\alpha/2} \right|^2 dv\, dt = 0 \ ,$$

with $\lim_{n\to\infty}(t_n^2 - t_n^1) = +\infty$. Then $(f(\cdot + t_n^1, \cdot))_{n\in\mathbb{N}}$ would converge for the same reasons as above to both f_∞^1 and f_∞^2, which are therefore equal. □

4.2 Large entropy regime

Assume now that

$$\limsup_{t\to+\infty} \kappa(t) = +\infty \ .$$

Let $\Phi(u) = u^\alpha$, $\alpha > 1$, or $\Phi(u) = u \log u$, $\alpha = 1$. We prove that the entropy, or to be precise the quantity $\int_0^1 \Phi\left(f\, e^{(\alpha-1)C}\right) e^{-C}\, dv$, is also unbounded.

Theorem 8 *Let $\alpha \geq 1$. Assume that $\limsup_{t\to+\infty} \kappa(t) = +\infty$ for some smooth global in time solution f of (1) with u, B and D depending nonlinearly on f according to (4), (5) and (6) for some smooth function m_2. Let $\rho = \int_0^1 f(t,v)\, dv \in (0,1]$. Assume that $E[\, f \,|\, g\,]$ is well defined and C^1 in t, with E, g, and κ defined by (7), (9) and (10). Then as $t \to +\infty$, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ with $\lim_{n\to+\infty} t_n = +\infty$ and $\lim_{n\to+\infty} \kappa(t_n) = +\infty$ such that as $n \to +\infty$,*

(i) *if $\alpha > 1$,*

$$\int_0^1 \Big(f(t_n,v)\Big)^\alpha e^{(\alpha-1)C(t_n,v)}\, dv \sim \rho \Big(\kappa(t_n)\Big)^{\alpha-1},$$

(ii) *if $\alpha = 1$,*

$$\int_0^1 f(t_n,v)\, \log f(t_n,v)\, dv \sim \rho\, \log\Big(\kappa(t_n)\Big) \ .$$

Proof. The result in case $\alpha > 1$ is an immediate consequence of Proposition 4, (i). In case $\alpha = 1$, the proof goes as follows. According to Proposition 4, (ii),

$$E[\, f|\, g\,] = \int_0^1 f\,(\log f + C)\, dv - \rho\, \log \kappa(t) \leq Constant \ , \quad \forall\, t \geq 0 \ .$$

On the other hand, $E[\, f|\, g\,] = \int_0^1 f\, \log\left(\frac{f}{g}\right) dv$ can be bounded from below by Jensen's inequality:

$$E[\, f|\, g\,] \geq \int_0^1 f\, dv\, \log\left(\frac{\int_0^1 f\, dv}{\int_0^1 g\, dv}\right) = \rho\log\rho - \rho\log\left(\int_0^1 e^{-C}\, dv\right) - \rho\, \log\kappa(t) \ .$$

Since C is uniformly bounded, this means that

$$\int_0^1 f\, \log f\, dv \sim \rho\, \log\kappa(t) \quad \text{if } \kappa(t) \to +\infty \ .$$

□

Remark 9 *Note that concentration has to occur in any L^p norm, $p > \alpha$: there exists some $\epsilon > 0$ such that for any given $\delta > 0$, there exists a measurable set A with $|A| < \delta$ and $\liminf_{t\to+\infty} \int_A f^p\,dv > \epsilon$. To prove this, let us interpolate $\|f\|_{L^q(A)}$ between $\|f\|_{L^1(A)} \leq \|f\|_{L^1(0,1)}$ and $\|f\|_{L^p(A)}$ by Hölder's inequality and take $q = \alpha$ if $\alpha > 1$. If $\alpha = 1$, take the logarithm:*

$$\log\left(\int_A f^q\,dv\right) \leq \frac{p-q}{p-1}\log\left(\int_A f\,dv\right) + \frac{q-1}{p-1}\log\left(\int_A f^p\,dv\right)\ , \quad 1 < q < p\ .$$

A differentiation w.r.t. q at $q = 1$ gives

$$\frac{\int_A f\log f\,dv}{\int_A f\,dv} \leq \frac{1}{p-1}\log\left(\frac{\int_A f^p\,dv}{\int_A f\,dv}\right)\ ,$$

which provides the result.

It is an open question to understand if such a regime, where κ is unbounded, exists. The growth of $\int_0^1 \Phi\left(f\,e^{(\alpha-1)C}\right)e^{-C}\,dv$ means that some concentration phenomenon is occuring at least in the norm measured by the entropy. Note that as far as we know, there is no Maximum Principle available for a solution of (1), even measured by some L^p norm (see Section 5 below). For a better understanding of the concentration, one should further study the entropy production term

$$\kappa^{1-\alpha}\int_0^1 D(v,t)\,e^{(\alpha-1)C}\left|\left(f^{\alpha/2}\right)' + \frac{\alpha}{2}\,C'\,f^{\alpha/2}\right|^2\,dv\ .$$

It has to be noted that neither the standard logarithmic Sobolev inequality nor convex Sobolev inequalities ($\alpha \in (1,2]$, see [1]) do apply since the usual condition for the entropy - entropy production method is not satisfied.

5 Further observations

We conclude with some calculations which shed additional light on the properties of solutions of (1),(4),(5),(6). First consider Eqn. (1) *without* boundary conditions, *i.e.,* $v \in \mathbb{R}$, and assume that f vanishes sufficiently fast at $\pm\infty$ (this is, of course, inconsistent with traffic dynamics, where negative speeds are excluded and there are natural upper speed limits). Let f be a solution of (1) and compute

$$\frac{d}{dt}\int (v-u)^2 f(t,v)\,dv = -2\,\frac{du}{dt}\int (v-u)f(t,v)\,dv + \int (v-u)^2\,\partial_t f(t,v)\,dv\ .$$

The first term on the right is zero by definition of $u = u(t)$. For the second term we use the equation and emphasize the dependence of B and D on $v - u(t)$ as specified, for example, in (5), (6). After an integration by parts

$$\int (v-u)^2\partial_t f(t,v)\,dv = 2\int B(v-u)\cdot(v-u)f\,dv - 2\int D(v-u)\cdot(v-u)\,\partial_v f\,dv\ . \quad (12)$$

Note that for B as defined by (5) we have $B(v-u(t))\cdot(v-u(t)) \leq 0$ with equality only if $v = u(t)$. Therefore, if $D \equiv 0$, we have $\frac{d}{dt}\int (v-u(t))^2 f(t,v)\,dv \leq 0$, with equality only if $f(t,v) = \rho\,\delta_{u(t)}(v)$. This is consistent with the (modeling) idea that in the absence of diffusion all the drivers will brake or accelerate towards the observed average speed. In the limit $t \to \infty$, $u(t)$ must converge towards a limit speed u_∞ which will in general depend on f_0.

The calculation leading to (12) remains the same if we include (zero flux) boundary conditions. Let us now consider $D \neq 0$ and $v \in \mathbb{R}$, *i.e.,* we ignore the boundary conditions. If we formally integrate by parts in the last term in (12) we obtain

$$2\int \Big[D'(v-u(t))\cdot(v-u(t)) + D(v-u(t))\Big]f(t,v)\,dv$$

(plus boundary contributions if $v \in [0,1]$).

Suppose for the moment that f is concentrated in a small neighborhood of u. Then, what happens to $\int (v-u(t))^2 f(t,v)\,dv$ depends on the behaviour (the sign!) of

$$B(x)\cdot x + D'(x)\cdot x + D(x) \tag{13}$$

with $x = v - u(t)$. As an example inspired by (5),(6), consider $D(x) = |x|^\gamma$ with $\gamma \in (0,+\infty)$ and $B(x) = -c\,x\,|x|$. This is actually a special case of the traffic flow model with $\delta = 0$, $c = C_B\rho = C_A(1-\rho)$, $m_2 \equiv 1$, $\sigma m_1(\rho) = 1$. The expression (13) becomes

$$G(x) := -c\,x^2(|x| - \beta\,|x|^{\gamma-2}) \tag{14}$$

where $\beta = (1+\gamma)/c$. Whether or not f will form concentrations at $u(t)$ depends on the sign of (14) near $x = 0$. If $G(x) > 0$ for $x \approx 0$, which is the case if $\gamma < 3$, then diffusion dominates and no concentrations occur. If $G(x) < 0$ for $x \approx 0$ then concentrations are a possibility.

We conclude with a remark on whether or not f can be expected to stay bounded (if f remains bounded, so will $\kappa(t)$, and the hypotheses of Theorem 6 will apply). Assume that, at time t, f assumes its maximum at v_0 (not on the boundary). By elementary calculations,

$$\partial_t f|_{v=v_0} + B'(v_0 - u(t)) f|_{v=v_0} = D(v_0 - u(t))\,\partial_v^2 f|_{v=v_0} \le 0\ .$$

Hence, if $v_0 \neq u(t)$, the maximum of f may grow, but its growth is controlled by $B'(v_0 - u(t))$ and damped by $D(v_0 - u(t))$. Indefinite growth is of course consistent with the formation of concentrations. We conjecture that the solutions will actually stay bounded for all time *unless* concentrations form.

Conclusion

We proved the global stability of the steady solutions of a nonlinear model of traffic flow with respect to the dynamics, assuming that a quantity related to the entropy (and most probably to the absence of concentrations) stays bounded. However, in contrast to the linear case [2], this does not discard the possibility of several steady states and the asymptotic state is clearly selected by the initial conditions.

Stop-and-go regimes, which are observed in traffic flow experiments, are therefore not predicted by the spatially homogeneous model corresponding to (4), (5) and (6), at least in the asymptotic regime. However, the multivalued fundamental diagrams in combination with density fluctuations (which are, of course, excluded *a fortiori* in the spatially homogeneous case) should produce stop-and-go phenomena for the full Vlasov-Fokker-Planck equation.

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