

# Non-linear stability of gaseous stars

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## Abstract

We construct steady states of the Euler-Poisson system with a barotropic equation of state as minimizers of a suitably defined energy functional. Their minimizing property implies the non-linear stability of such states against general, i.e., not necessarily spherically symmetric perturbations. The mathematical approach is based on previous stability results for the Vlasov-Poisson system by Y. Guo and the author, exploiting the energy-Casimir technique. The analysis is conditional in the sense that it *assumes* the existence of solutions to the initial value problem for the Euler-Poisson system which preserve mass and energy. The relation between the Euler-Poisson and the Vlasov-Poisson system in this context is also explored.

## 1 Introduction

Consider a self-gravitating fluid ball in  $\mathbb{R}^3$  where the fluid has mass density  $\rho = \rho(t, x) \geq 0$  and velocity field  $u = u(t, x) \in \mathbb{R}^3$ ;  $t \geq 0$  denotes time and  $x \in \mathbb{R}^3$  position. In the simplest case such a system obeys the Euler-Poisson equations

$$\partial_t \rho + \nabla \cdot (\rho u) = 0, \tag{1.1}$$

$$\rho \partial_t u + \rho (u \cdot \nabla) u = -\nabla p - \rho \nabla V, \tag{1.2}$$

$$\Delta V = 4\pi \rho, \quad \lim_{|x| \rightarrow \infty} V(t, x) = 0, \tag{1.3}$$

where  $V = V(t, x)$  denotes the self-consistent gravitational potential and  $p = p(t, x)$  is the pressure in the fluid. To close this system an equation of state needs to be supplied which relates the pressure and the density:

$$p = P(\rho). \quad (1.4)$$

An equation of state where the pressure does not depend on the temperature or specific entropy is often referred to as barotropic. More sophisticated models for a star would contain equations for the temperature and for the thermonuclear processes within the star, but in astrophysics the present model is often used as a simple description of a gaseous star, in particular if—as in the present note—the issue is the stability of steady states since this problem is non-trivial already for this simple model.

To approach this problem we define the energy of a state  $(\rho, u)$  as

$$\mathcal{H}(\rho, u) := \frac{1}{2} \int |u|^2 \rho dx + \int \Phi(\rho) dx - \frac{1}{2} \int \frac{\rho(x)\rho(y)}{|x-y|} dy dx. \quad (1.5)$$

Here  $\Phi$  is defined in such a way that for  $\rho > 0$ ,

$$P'(\rho) = \rho \Phi''(\rho). \quad (1.6)$$

Under appropriate assumptions on  $\Phi$  or  $P$  respectively, which include—but are much more general than—equations of state of the form  $P(\rho) = c\rho^\gamma$  with  $\gamma > 4/3$ , this functional has a minimizer under the mass constraint

$$\int \rho dx = M$$

with  $M > 0$  prescribed. This follows from a result of the author established in [23]. Such a minimizer is a spherically symmetric steady state of the Euler-Poisson system with vanishing velocity field and as such is referred to as a static solution of the system, as opposed to the more general concept of a stationary solution, which is also time-independent but may have non-vanishing velocity field. From the fact that the steady state minimizes the energy one can deduce a rigorous, non-linear stability result. Since in order to solve the variational problem stated above we need to make no symmetry assumption we obtain stability against general, i.e., not necessarily spherically symmetric perturbations, even though the steady state obtained as an energy minimizer must a posteriori be spherically symmetric. An essential point for the stability analysis is that both the energy defined in (1.5) and the total mass  $\int \rho dx$

are conserved along solutions of the time dependent problem. Since the initial value problem for the Euler-Poisson system is far from being completely understood and one has to expect the occurrence of shocks or gravitational collapse our stability result is conditional in the following sense: As long as a solution to the initial value problem exists and preserves mass and energy it satisfies the stability estimate.

The paper proceeds as follows: In the next section we study the variational problem stated above. Then we show that a minimizer is a spherically symmetric, static solution. In Section 3 we carry out the stability analysis. In a final section we discuss the relation of the present result to those obtained by Y. Guo and the author for the Vlasov-Poisson system

$$\partial_t f + v \cdot \nabla_x f - \nabla V \cdot \nabla_v f = 0,$$

$$\Delta V = 4\pi \rho, \quad \lim_{|x| \rightarrow \infty} V(t, x) = 0,$$

$$\rho(t, x) = \int f(t, x, v) dv.$$

Here the dynamic variable is the number density in phase space,  $f = f(t, x, v)$ , of an ensemble of massive particles with spatial density  $\rho$ , interacting by the gravitational potential  $V$  which the particles create collectively;  $v \in \mathbb{R}^3$  denotes the momentum or velocity coordinate in phase space. In this model collisions among the particles are neglected which is appropriate if one is describing a galaxy or a globular cluster. In [6, 7, 8, 9, 10, 20, 22] a variational approach to the question of the stability of steady states of this system was developed, which relied on the minimization of appropriately defined energy-Casimir functionals. In [23] the author observed that these functionals, which act on phase space densities  $f = f(x, v)$ , can in a natural way be reduced to functionals acting on spatial densities  $\rho = \rho(x)$ , and he studied the variational problem for these reduced functionals, without realizing that this yields the stability result for the Euler-Poisson problem which we discuss here. Incidentally, the close relation between the stability of barotropic stars and of stellar systems was already observed on a formal level in the astrophysics literature and is often referred to as Antonov's First Law, cf. [1, 5.2 (b)].

We conclude this introduction with some additional references to the literature. A lot of background on the physics of stars can be found in [2] and [13]. An excellent and broad overview of mathematical results for hydrodynamical models of gaseous stars is given in [4] which also contains many

further references. Our variational approach is related to the concentration-compactness principle due to P.-L. Lions [17]. The use of energy-Casimir functionals for questions of stability was discussed in a very broad context in [12]. As aimed more specifically to the existence and stability properties of steady states for gaseous stars we mention [3, 16] and the references there.

## 2 Energy minimizers and steady states

If we minimize the energy  $\mathcal{H}$  as defined in (1.5) over all “relevant” states  $(\rho, u)$  then a minimizer will clearly satisfy  $u = 0$ . Thus we study the following variational problem: For a density  $\rho$  we define the induced potential

$$V_\rho(x) := - \int \frac{\rho(y)}{|x-y|} dy,$$

and the potential energy

$$E_{\text{pot}}(\rho) := -\frac{1}{2} \iint \frac{\rho(x)\rho(y)}{|x-y|} dy dx = \frac{1}{2} \int \rho V_\rho dx = -\frac{1}{8\pi} \int |\nabla V_\rho|^2 dx. \quad (2.1)$$

We want to minimize the functional

$$\mathcal{H}_r(\rho) := \int \Phi(\rho(x)) dx + E_{\text{pot}}(\rho) \quad (2.2)$$

over the constraint set

$$\mathcal{F}_M := \left\{ \rho \in L^1(\mathbb{R}^3) \mid \rho \geq 0, \int \Phi(\rho) < \infty, \int \rho = M \right\} \quad (2.3)$$

for  $M > 0$  prescribed and  $\Phi$  satisfying the following

**Assumptions on  $\Phi$ :**  $\Phi \in C^1([0, \infty[)$ ,  $\Phi(0) = 0 = \Phi'(0)$ , and

( $\Phi 1$ )  $\Phi$  is strictly convex.

( $\Phi 2$ )  $\Phi(\rho) \geq C\rho^{1+1/n}$  for  $\rho \geq 0$  large, with  $0 < n < 3$ ,

( $\Phi 3$ )  $\Phi(\rho) \leq C\rho^{1+1/n'}$  for  $\rho \geq 0$  small, with  $0 < n' < 3$ .

Note that  $\rho \in L^1 \cap L^{4/3}(\mathbb{R}^3)$  and hence  $V_\rho \in L^p(\mathbb{R}^3)$ ,  $\nabla V_\rho \in L^q(\mathbb{R}^3)$  for  $\rho \in \mathcal{F}_M$  and any  $p \in ]3, 12]$  and  $q \in ]3/2, 12/5]$  so that the potential energy is well defined. The following result was proved earlier by the author, cf. [23, Thm. 3.1]; it is included here for easier reference.

**Theorem 1** *The functional  $\mathcal{H}_r$  is bounded from below on  $\mathcal{F}_M$  with  $\inf_{\mathcal{F}_M} \mathcal{H}_r < 0$ . Let  $(\rho_i) \subset \mathcal{F}_M$  be a minimizing sequence of  $\mathcal{H}_r$ . Then there exists a sequence of shift vectors  $(a_i) \subset \mathbb{R}^3$  and a subsequence, again denoted by  $(\rho_i)$ , such that for any  $\epsilon > 0$  there exists  $R > 0$  with*

$$\int_{a_i + B_R} \rho_i(x) dx \geq M - \epsilon, \quad i \in \mathbb{N},$$

$$T\rho_i := \rho_i(\cdot + a_i) \rightharpoonup \rho_0 \text{ weakly in } L^{1+1/n}(\mathbb{R}^3), \quad i \rightarrow \infty,$$

and

$$\int_{B_R} \rho_0 \geq M - \epsilon.$$

Finally,

$$\nabla V_{T\rho_i} \rightarrow \nabla V_0 \text{ strongly in } L^2(\mathbb{R}^3), \quad i \rightarrow \infty,$$

where  $V_0 = V_{\rho_0}$ , and  $\rho_0 \in \mathcal{F}_M$  is a minimizer of  $\mathcal{H}_r$ .

Here and in the following we denote for  $R > 0$ ,

$$B_R := \{x \in \mathbb{R}^3 \mid |x| \leq R\}.$$

**Remark 1.** As noted in the introduction, the function  $\Phi$  appearing in the definition of the energy and the function  $P$  defining the equation of state (1.4) have to satisfy the relation (1.6). If we think of  $P$  as the basic, given quantity then we define

$$\Phi(\rho) := \int_0^\rho \int_0^\sigma \frac{P'(\tau)}{\tau} d\tau d\sigma, \quad \rho \geq 0,$$

and this function satisfies the assumptions  $(\Phi 1)$ ,  $(\Phi 2)$ ,  $(\Phi 3)$  provided  $P'(\tau)/\tau$  is integrable on  $]0, \infty[$  and the assumptions

$$(P1) \quad P' > 0,$$

$$(P2) \quad P'(\tau) \geq C\tau^{1/n}, \quad \tau \geq 0 \text{ large, with } 0 < n < 3,$$

$$(P3) \quad P'(\tau) \leq C\tau^{1/n'}, \quad \tau \geq 0 \text{ small, with } 0 < n' < 3$$

hold. This is the case in particular for polytropic equations of state

$$P(\rho) = c\rho^\gamma$$

with  $c > 0$  and  $\gamma > 4/3$ , in which case

$$\Phi(\rho) = \frac{c}{\gamma - 1} \rho^\gamma.$$

Equations of state where the dependence of the pressure on the density is different for large and for small values of the density are motivated from a physics point of view.

**Remark 2.** The need for the spatial shifts in Theorem 1 is not technical since without them the assertion of the theorem is false: Given a minimizer  $\rho_0$  the fact that the functional  $\mathcal{H}_r$  is translation invariant implies that by shifting  $\rho_0$  off to infinity one obtains a minimizing sequence which converges weakly to zero and not to a minimizer.

Now that a minimizer is obtained we show that it is a steady state of the Euler-Poisson system; in order to avoid technical discussions concerning regularity let us now assume in addition to the above that  $\Phi \in C^2(]0, \infty[)$  which means  $P \in C^1(]0, \infty[)$ .

**Theorem 2** *Suppose that  $\rho_0 \in \mathcal{F}_M$  is a minimizer of the functional  $\mathcal{H}_r$  with induced potential  $V_0$ . Then there exists a Lagrange multiplier  $E_0 \leq 0$  such that*

$$\rho_0 = \begin{cases} (\Phi')^{-1}(E_0 - V_0), & V_0 < E_0, \\ 0, & V_0 \geq E_0. \end{cases}$$

*The functions  $\rho_0$  and  $V_0$  are spherically symmetric with respect to some point in  $\mathbb{R}^3$ , as functions of the radial variable  $\rho_0$  is decreasing and  $V_0$  increasing,  $\rho_0 \in C(\mathbb{R}^3)$  with finite mass  $M$ ,  $V_0 \in C^2(\mathbb{R}^3)$  with  $\lim_{|x| \rightarrow \infty} V_0(x) = 0$ , and  $(\rho_0, u_0 = 0)$  is a time-independent solution of the Euler-Poisson system with equation of state*

$$P(\rho) = \int_0^\rho \sigma \Phi''(\sigma) d\sigma.$$

*If  $\Phi'(\rho) \leq C\rho^{1/n'}$  for  $\rho \geq 0$  small then  $E_0 < 0$  and  $\rho_0$  has compact support.*

The assumptions on  $\Phi$  imply that  $\Phi'$  is invertible on  $[0, \infty[$ . The growth condition on  $\Phi'$  implies the condition  $(\Phi 3)$  but is not equivalent to it.

**Proof of Theorem 2.** We start by deriving the Euler-Lagrange equation for the variational problem. Let  $\rho_0 \in \mathcal{F}_M$  be a minimizer with induced potential  $V_0$ . For  $\epsilon > 0$  define

$$S_\epsilon := \{x \in \mathbb{R}^3 \mid \epsilon \leq \rho_0(x) \leq 1/\epsilon\};$$

think of  $\rho_0$  as a pointwise defined representative of the minimizer. For a test function  $w \in L^\infty(\mathbb{R}^3)$  which has compact support and is non-negative on  $\mathbb{R}^3 \setminus S_\epsilon$  define for  $\tau \geq 0$  small,

$$\rho_\tau := \rho_0 + \tau w - \tau \frac{\int w dy}{\text{vol} S_\epsilon} \mathbf{1}_{S_\epsilon},$$

where  $\mathbf{1}_{S_\epsilon}$  denotes the indicator function of the set  $S_\epsilon$ . Then  $\rho_\tau \geq 0$  and  $\int \rho_\tau = M$  so that  $\rho_\tau \in \mathcal{F}_M$  for  $\tau \geq 0$  small. Since  $\rho_0$  is a minimizer of  $\mathcal{H}_r$ ,

$$0 \leq \mathcal{H}_r(\rho_\tau) - \mathcal{H}_r(\rho_0) = \tau \int (\Phi'(\rho_0) + V_0) \left( w - \frac{\int w dy}{\text{vol} S_\epsilon} \mathbf{1}_{S_\epsilon} \right) dx + o(\tau).$$

Hence the coefficient of  $\tau$  in this estimate must be non-negative, which we can rewrite in the form

$$\int \left[ \Phi'(\rho_0) + V_0 - \frac{1}{\text{vol} S_\epsilon} \left( \int_{S_\epsilon} (\Phi'(\rho_0) + V_0) dy \right) \right] w dx \geq 0.$$

This holds for all test functions  $w$  as specified above, and hence  $\Phi'(\rho_0) + V_0 = E_\epsilon$  on  $S_\epsilon$  and  $\Phi'(\rho_0) + V_0 \geq E_\epsilon$  on  $\mathbb{R}^3 \setminus S_\epsilon$  for all  $\epsilon > 0$  small enough. Here  $E_\epsilon$  is some constant which by the first relation must be independent of  $\epsilon$ , and taking  $\epsilon \rightarrow 0$  proves the relation between  $\rho_0$  and  $V_0$ .

The symmetry assertion follows by a rearrangement argument: Let  $\rho_0^*$  denote the symmetric decreasing rearrangement of  $\rho_0$ . Then

$$\int \Phi(\rho_0) = \int \Phi(\rho_0^*), \quad E_{\text{pot}}(\rho_0) \geq E_{\text{pot}}(\rho_0^*),$$

and  $\rho_0^* \in \mathcal{F}_M$ , cf. [15, Thms. 3.7, 3.9]. Since  $\rho_0$  is a minimizer of  $\mathcal{H}_r$  equality must hold in the estimate for the potential energy which implies that  $\rho_0$  must be spherically symmetric with respect to some point in  $\mathbb{R}^3$ , and decreasing as a function of the radial variable. Alternatively, one could use [5, Thm. 4] to conclude the spherical symmetry.

We now continue essentially as in the proof of [22, Thm. 3]. First we observe that by assumptions  $(\Phi 1)$  and  $(\Phi 2)$  and the mean value theorem

$$\Phi'(\rho) \geq \Phi'(\sigma) = \frac{\Phi(\rho) - \Phi(0)}{\rho - 0} \geq C \rho^{1/n}$$

for all  $\rho$  large, with some intermediate value  $0 \leq \sigma \leq \rho$ . Hence

$$\rho_0(x) \leq C(1 + (E_0 - V_0(x))^n)$$

and since  $V_0 \in L^{12}(\mathbb{R}^3)$  and  $n < 3$  we have  $\rho_0 \in L^1 \cap L^4(\mathbb{R}^3)$ . For any  $R > 1$  we split the convolution integral for  $V_0$  according to  $|x - y| < 1/R$ ,  $1/R \leq |x - y| < R$ , and  $|x - y| \geq R$  to obtain

$$-V_0(x) \leq C \|\rho_0\|_4 \left( \int_0^{1/R} r^{2-4/3} dr \right)^{3/4} + R \int_{|y| \geq |x| - R} \rho_0(y) dy + \frac{M}{R}, \quad |x| \geq R.$$

This implies that  $V_0 \in L^\infty(\mathbb{R}^3)$  with  $V_0(x) \rightarrow 0$ ,  $|x| \rightarrow \infty$ . The asserted regularity of  $\rho_0$  and  $V_0$  now follows from the relation between these two quantities and Sobolev's embedding theorem.

The limiting behavior of  $V_0$  and the relation between  $\rho_0$  and  $V_0$  implies that  $E_0 \leq 0$ , since otherwise  $\rho_0(x) \geq (\Phi')^{-1}(E_0/2) > 0$  for  $|x|$  large which would contradict the integrability of  $\rho_0$ .

Next we check that  $(\rho_0, u_0 = 0)$  solves the stationary Euler-Poisson system with the asserted equation of state. Since by construction  $V_0$  is the potential induced by  $\rho_0$  all that needs to be checked is the Euler equation which takes the form

$$\nabla p_0 + \rho_0 \nabla V_0 = 0.$$

Now  $\Phi'(\rho_0) = E_0 - V_0$  on  $\text{supp } \rho_0$ . Taking the gradient of this relation proves the assertion, provided  $P' = \Phi'' \rho$ .

It remains to show that under the additional growth condition on  $\Phi'$  the density is compactly supported and the cut-off energy  $E_0$  strictly negative. Since  $V_0$  increases to zero at spatial infinity the relation between  $\rho_0$  and  $V_0$  implies that  $\rho_0$  has compact support provided  $E_0 < 0$ . If we assume that  $E_0 = 0$  then the additional assumption on  $\Phi'$  implies that

$$\rho_0(x) \geq C(-V_0(x))^{n'}$$

for  $|x|$  large, i.e.,  $\rho_0$  small. But a simple expansion of the Green's function in the convolution formula for  $V_0$  and the fact that  $\int \rho_0 = M$  show that  $-V_0(x) \geq M/(3|x|)$  for  $|x|$  large. Inserting this into the estimate for  $\rho_0$  from below yields a contradiction to the integrability of  $\rho_0$ . Hence  $E_0 < 0$ , and the proof is complete.  $\square$

In view of the stability result discussed in the next section it would be desirable to know that for fixed  $M$  there is up to spatial shifts a unique minimizer of  $\mathcal{H}_r$  in  $\mathcal{F}_M$  or at least that the minimizers are isolated up to spatial shifts. Numerical evidence obtained by solving the spherically symmetric Poisson problem  $\Delta V_0 = 4\pi \rho_0 = 4\pi(\Phi')^{-1}(E_0 - V_0)$  seems to indicate that the



minimizers are in general not unique but are isolated. An example where the minimizer is unique is provided by the polytropic equation of state:

**Remark 3.** Let  $P(\rho) = c\rho^\gamma$ , i.e.,  $\Phi(\rho) = \frac{c}{\gamma-1}\rho^\gamma$  with  $\gamma > 4/3$ . Then for every  $M > 0$  there exists up to spatial shifts exactly one minimizer  $\rho_0 \in \mathcal{F}_M$  of  $\mathcal{H}_r$ . This follows from the fact that  $E_0 - V_0$  solves the Emden-Fowler equation

$$\frac{1}{r^2}(r^2 z')' = -cz_+^n, \quad r > 0,$$

where  $1 + 1/n = \gamma$  and  $z_+$  denotes the positive part of  $z$ . All solutions of this equation which are regular at the center are related by a scaling transformation, and prescribing the mass fixes the corresponding scaling parameter. For more details cf. [22, Thm. 3 (b)].

### 3 The stability analysis

We start by expanding the total energy  $\mathcal{H}$  about a minimizer  $\rho_0$ : For  $\rho \in \mathcal{F}_M$  and  $u$  such that  $\int |u|^2 \rho < \infty$  we have

$$\mathcal{H}(\rho, u) - \mathcal{H}(\rho_0, 0) = d(\rho, \rho_0) - \frac{1}{8\pi} \|\nabla V_\rho - \nabla V_0\|_2^2 + \frac{1}{2} \int |u|^2 \rho dx \quad (3.1)$$

where

$$d(\rho, \rho_0) := \int [\Phi(\rho) - \Phi(\rho_0) + (V_0 - E_0)(\rho - \rho_0)] dx;$$

we are allowed to insert the term  $E_0(\rho - \rho_0)$  into  $d$  since  $\int \rho = \int \rho_0$ . The fact that  $\Phi$  is convex and the relation between  $\rho_0$  and  $V_0$  established in Theorem 2 imply that for all  $\rho \in \mathcal{F}_M$ ,

$$d(\rho, \rho_0) \geq 0.$$

Moreover,  $d(\rho, \rho_0) = 0$  only if  $\rho = \rho_0$ , and if one assumes in addition that  $\Phi''$  is bounded away from zero then

$$d(\rho, \rho_0) \geq C \|\rho - \rho_0\|_2^2, \quad \rho \in \mathcal{F}_M.$$

What is irritating about (3.1) is the minus sign in front of the difference term for the gravitational fields, but this term converges to zero along minimizing sequences, cf. Theorem 1.

Our stability analysis relies on the fact that the mass and the total energy  $\mathcal{H}$  defined in (1.5) are conserved along solutions of the time-dependent Euler-Poisson system. Formally, conservation of mass follows from the continuity

equation (1.1), and conservation of energy follows by a straight forward computation from all three equations (1.1), (1.2), (1.3) together, provided the relation (1.6) between the equation of state  $p = P(\rho)$  and the function  $\Phi$  which appears in the energy holds. However, from a rigorous mathematical point of view insisting on these conservation laws amounts to an *assumption* on the solutions of the initial value problem. This is reflected in the statement of our stability result.

**Theorem 3** *Let  $\rho_0 \in \mathcal{F}_M$  be a minimizer of  $\mathcal{H}_r$  with induced potential  $V_0$ , and assume that it is unique up to spatial shifts. Then for every  $\epsilon > 0$  there is a  $\delta > 0$  such that the following holds: For every solution  $t \mapsto (\rho(t), u(t))$  of the Euler-Poisson system with  $\rho(0) \in \mathcal{F}_M$  which preserves mass and energy the initial estimate*

$$d(\rho(0), \rho_0) + \frac{1}{8\pi} \|\nabla V_{\rho(0)} - \nabla V_0\|_2^2 + \frac{1}{2} \int |u(0)|^2 \rho(0) dx < \delta$$

*implies that for every  $t \geq 0$  for which the solution still exists there is a shift vector  $a \in \mathbb{R}^3$  such that*

$$d(\rho(t), T^a \rho_0) + \frac{1}{8\pi} \|\nabla V_{\rho(t)} - \nabla V_{T^a \rho_0}\|_2^2 + \frac{1}{2} \int |u(t)|^2 \rho(t) dx < \epsilon;$$

*here  $T^a \rho := \rho(\cdot + a)$ .*

**Proof.** Assume the assertion were false. Then there exist  $\epsilon > 0$ ,  $t_n > 0$ , and initial data  $\rho_n(0) \in \mathcal{F}_M$  and  $u_n(0)$  such that for all  $n \in \mathbb{N}$ ,

$$d(\rho_n(0), \rho_0) + \frac{1}{8\pi} \|\nabla V_{\rho_n(0)} - \nabla V_0\|_2^2 + \frac{1}{2} \int |u_n(0)|^2 \rho_n(0) dx < \frac{1}{n} \quad (3.2)$$

but for any  $a \in \mathbb{R}^3$ ,

$$d(\rho_n(t_n), T^a \rho_0) + \frac{1}{8\pi} \|\nabla V_{\rho_n(t_n)} - \nabla V_{T^a \rho_0}\|_2^2 + \frac{1}{2} \int |u_n(t_n)|^2 \rho_n(t_n) dx \geq \epsilon. \quad (3.3)$$

By (3.2) and (3.1),

$$\lim_{n \rightarrow \infty} \mathcal{H}(\rho_n(0), u_n(0)) = \mathcal{H}(\rho_0, 0) = \mathcal{H}_r(\rho_0).$$

Since by assumption the solution  $t \mapsto (\rho_n(t), u_n(t))$  conserves energy,

$$\limsup_{n \rightarrow \infty} \mathcal{H}_r(\rho_n(t_n)) \leq \lim_{n \rightarrow \infty} \mathcal{H}(\rho_n(t_n), u_n(t_n)) = \mathcal{H}_r(\rho_0),$$

and since it conserves mass,  $(\rho_n(t_n)) \subset \mathcal{F}_M$  is a minimizing sequence for  $\mathcal{H}_r$ . By Theorem 1 there exists a sequence  $(a_n) \subset \mathbb{R}^3$  such that up to a subsequence,

$$\|\nabla V_{\rho_n(t_n)} - \nabla V_{T^{a_n} \rho_0}\|_2 \rightarrow 0; \quad (3.4)$$

at this point we used the uniqueness of the minimizer. Note also that for any  $\rho \in \mathcal{F}_M$  and  $a \in \mathbb{R}^3$ ,

$$\|\nabla V_{T^a \rho} - \nabla V_{\rho_0}\|_2 = \|\nabla V_{\rho} - \nabla V_{T^{-a} \rho_0}\|_2, \quad d(T^a \rho, \rho_0) = d(\rho, T^{-a} \rho_0).$$

Since  $\lim_{n \rightarrow \infty} \mathcal{H}(\rho_n(t_n), u_n(t_n)) = \mathcal{H}_r(\rho_0) = \mathcal{H}_r(T^{a_n} \rho_0)$  we conclude by (3.4) and (3.1) that

$$d(\rho_n(t_n), T^{a_n} \rho_0) + \frac{1}{2} \int |u_n(t_n)|^2 \rho_n(t_n) dx \rightarrow 0, \quad n \rightarrow \infty,$$

a contradiction to (3.3). □

**Remark 4.** Without the uniqueness assumption for the minimizer we obtain a stability result of the following type: Let  $\mathcal{M}_M \subset \mathcal{F}_M$  denote the set of all minimizers of  $\mathcal{H}_r$  in  $\mathcal{F}_M$ . Then for every  $\epsilon > 0$  there is a  $\delta > 0$  such that for any solution with  $\rho(0) \in \mathcal{F}_M$  the initial estimate

$$\inf_{\rho_0 \in \mathcal{M}_M} \left[ d(\rho(0), \rho_0) + \frac{1}{8\pi} \|\nabla V_{\rho(0)} - \nabla V_0\|_2^2 + \frac{1}{2} \int |u(0)|^2 \rho(0) dx \right] < \delta$$

implies that

$$\inf_{\rho_0 \in \mathcal{M}_M} \left[ d(\rho(t), \rho_0) + \frac{1}{8\pi} \|\nabla V_{\rho(t)} - \nabla V_{\rho_0}\|_2^2 + \frac{1}{2} \int |u(t)|^2 \rho(t) dx \right] < \epsilon \quad (3.5)$$

as long as the solution  $(\rho(t), u(t))$  exists and preserves mass and energy. The proof of this assertion follows exactly the same line of reasoning as the proof of Theorem 3.

Assume now that the minimizer  $\rho_0$  is not necessarily unique, but isolated up to spatial shifts, that is to say,

$$\delta_0 := \inf \left\{ \|\nabla V_{\rho_0} - \nabla V_{\tilde{\rho}_0}\|_2 \mid \tilde{\rho}_0 \in \mathcal{M}_M \setminus \{T^a \rho_0 \mid a \in \mathbb{R}^3\} \right\} > 0.$$

In this case the assertion of the theorem again holds, provided the solution of the Euler-Poisson system is sufficiently continuous in  $t$  so that it cannot jump from one minimizer to the next, more precisely: Let  $\epsilon > 0$  arbitrary. In

order to find the corresponding  $\delta$  we can without loss of generality assume that  $\epsilon < \delta_0/4$ . Now choose  $\delta > 0$  according to the first part of the present remark so that (3.5) holds, without loss of generality  $\delta < \epsilon$ , and let  $\rho(0) \in \mathcal{F}_M$  and  $u(0)$  be such that

$$d(\rho(0), \rho_0) + \frac{1}{8\pi} \|\nabla V_{\rho(0)} - \nabla V_0\|_2^2 + \frac{1}{2} \int |u(0)|^2 \rho(0) dx < \delta.$$

The required continuity assumption on the corresponding solution is that

$$h(t, a) := d(\rho(t), T^a \rho_0) + \frac{1}{8\pi} \|\nabla V_{\rho(t)} - \nabla V_{T^a \rho_0}\|_2^2 + \frac{1}{2} \int |u(t)|^2 \rho(t) dx$$

is continuous, and indeed, that  $\inf_{a \in \mathbb{R}^3} h(t, a)$  is continuous. Now assume that there exists  $t > 0$  such that

$$\inf_{a \in \mathbb{R}^3} h(t, a) \geq \epsilon.$$

Since at time zero the left hand side is less than  $\epsilon$  there exists some  $t^* > 0$  where

$$\inf_{a \in \mathbb{R}^3} h(t^*, a) = \epsilon. \quad (3.6)$$

On the other hand, the first part of the present remark provides some  $\rho_0^* \in \mathcal{M}_M$  such that

$$d(\rho(t^*), \rho_0^*) + \frac{1}{8\pi} \|\nabla V_{\rho(t^*)} - \nabla V_{\rho_0^*}\|_2^2 + \frac{1}{2} \int |u(t^*)|^2 \rho(t^*) dx < \epsilon \leq \frac{\delta_0}{4}. \quad (3.7)$$

By (3.6) and (3.7) together with the non-negativity of  $d$ ,

$$\frac{1}{8\pi} \|\nabla V_{\rho_0} - \nabla V_{\rho_0^*}\|_2^2 \leq \frac{\delta_0}{2},$$

and by the definition of  $\delta_0$  there must exist some  $a^* \in \mathbb{R}^3$  such that  $\rho_0^* = T^{a^*} \rho_0$ . But this means that (3.6) contradicts (3.7), and the argument is complete.

**Remark 5.** The restriction  $\rho(0) \in \mathcal{F}_M$  for the perturbed initial data is acceptable from a physics point of view: A small perturbation of a given star, say by the gravitational pull of some outside object, results in a perturbed state with the same mass. However, such a perturbation will hardly be spherically symmetric so that it is important that no such restriction is necessary in our stability result. On the other hand, if we restrict ourselves to spherically symmetric perturbations then the spatial shifts in the statement of the stability result are no longer necessary. Their need arises from the fact that a given steady state can be given a uniform velocity in one direction so that it drifts off and the distance of the perturbed state from the original one grows linearly in  $t$ , no matter how small the initial perturbation was.

## 4 Fluid models versus kinetic models

Obviously, the hard part in the analysis, if any, is the proof of Theorem 1. The essential arguments for that theorem as well as its exploitation for stability questions arose from the investigation of the stability of galaxies by Y. Guo and the author. In typical galaxies even stars which are spatially close to each other can have very different velocities, and hence galaxies are modeled not by fluid equations but by kinetic equations, i.e., by the Vlasov-Poisson system which was stated in the introduction. On the level of existence and regularity of solutions to the initial value problem the Vlasov-Poisson system is very different from the Euler-Poisson system: Continuously differentiable and compactly supported initial data for  $f$  launch continuously differentiable solutions which are global in time, cf. [19, 18, 25], and neither shocks nor a gravitational collapse can occur. On the other hand, as to the existence and stability of stationary solutions both systems seem to be intimately related, and in this last section we want to comment on this relation.

In [6, 7, 8, 9, 20, 22] steady states of the Vlasov-Poisson system were obtained as minimizers of an energy-Casimir functional

$$\mathcal{H}_C(f) = \iint Q(f(x, v)) dv dx + \frac{1}{2} \iint |v|^2 f(x, v) dv dx - \frac{1}{2} \iint \frac{\rho_f(x) \rho_f(y)}{|x - y|} dx dy$$

on the constraint set

$$\mathcal{G}_M := \left\{ f \in L^1(\mathbb{R}^6) \mid f \geq 0, \mathcal{C}(f) + E_{\text{kin}}(f) < \infty, \iint f dv dx = M \right\},$$

where  $\mathcal{C}(f)$  and  $E_{\text{kin}}(f)$  denote the first and second term in the energy-Casimir functional respectively. Such minimizers exist, satisfy assertions analogous to Theorems 1 and 2, and are non-linearly stable, provided  $Q$  satisfies assumptions which are exactly parallel to the ones for  $\Phi$ , except that the parameters  $0 < n, n' < 3$  have to be replaced by parameters  $0 < k, k' < 3/2$ .

In [23] it was observed that there is a one-to-one correspondence between the minimizers of the energy-Casimir functional  $\mathcal{H}_C$  on the “kinetic” set  $\mathcal{G}_M$  and the minimizers of the “reduced” functional  $\mathcal{H}_r$  on the set  $\mathcal{F}_M$ : If  $\rho_0$  is a minimizer of the latter functional with induced potential  $V_0$  then

$$f_0 := \begin{cases} (Q')^{-1}(E_0 - E), & E < E_0, \\ 0, & E \geq E_0 \end{cases}$$

is a minimizer of  $\mathcal{H}_C$ , where the particle energy  $E$  is defined as

$$E = E(x, v) := \frac{1}{2}|v|^2 + V_0(x).$$

On the other hand, if  $f_0$  is a minimizer of  $\mathcal{H}_C$  then the induced spatial density  $\rho_0 = \int f_0 dv$  minimizes  $\mathcal{H}_r$ . Of course, in order for this correspondence to be true the functions  $Q$  and  $\Phi$  have to be in the proper relation:

$$\Phi(r) = \inf_{g \in \mathcal{G}_r} \int \left( \frac{1}{2}|v|^2 g(v) + Q(g(v)) \right) dv$$

where for  $r \geq 0$ ,

$$\mathcal{G}_r := \left\{ g \in L^1(\mathbb{R}^3) \mid g \geq 0, \int \left( \frac{1}{2}|v|^2 g(v) + Q(g(v)) \right) dv < \infty, \int g(v) dv = r \right\}.$$

With this relation it follows that for every  $f \in \mathcal{G}_M$ ,

$$\mathcal{H}_C(f) \geq \mathcal{H}_r(\rho_f),$$

and if  $f = f_0$  is a minimizer of  $\mathcal{H}_C$  over  $\mathcal{F}_M$  then equality holds; one first minimizes over all  $f$  which give the same spatial density  $\rho$  and then minimizes over the latter. Indeed, there is a more explicit relation between  $Q$  and  $\Phi$  via their Legendre transforms; for a function  $h: \mathbb{R} \rightarrow ]-\infty, \infty]$  we denote by

$$h^*(\lambda) := \sup_{r \in \mathbb{R}} (\lambda r - h(r))$$

its Legendre transform. If we extend  $\Phi$  and  $Q$  to  $]-\infty, 0[$  by  $+\infty$  then

$$\Phi^*(\lambda) = \int Q^* \left( \lambda - \frac{1}{2}|v|^2 \right) dv,$$

for  $\lambda \in \mathbb{R}$ , and it can be seen that the assumptions on  $Q$  translate into the corresponding ones for  $\Phi$  where the relevant exponents are related by  $n = k + 3/2$ . These assertions are established in [23], cf. also [27].

If one is interested only in the existence of spherically symmetric steady states and not their stability the relation becomes even more direct: Suppose  $f_0$  is such a steady state of the Vlasov-Poisson system and define its radial and tangential pressures as functions of the radial variable  $r = |x|$  by

$$p_0(r) = \int \left( \frac{x \cdot v}{r} \right)^2 f_0(x, v) dv, \quad p_0^T(r) = \frac{1}{2} \int \left| \frac{x \times v}{r} \right|^2 f_0(x, v) dv.$$

Then a simple computation using the Vlasov equation implies the relation

$$p'_0 = \frac{2}{r}(p_0^T - p_0) - \rho_0 V'_0, \quad (4.1)$$

the Tolman-Oppenheimer-Volkov equation. If we make the isotropic ansatz

$$f_0(x, v) = \phi(E(x, v)) \quad (4.2)$$

with  $\phi \geq 0$  prescribed and the particle energy  $E$  given in terms of the potential  $V_0$  as above then the pressure is isotropic,  $p_0^T = p_0$ , and (4.1) reduces to the static Euler equation  $p'_0 + \rho_0 V'_0 = 0$ . Moreover,

$$\begin{aligned} \rho_0(r) &= 2^{5/2} \pi \int_{V_0(r)}^{\infty} \phi(E) (E - V_0(r))^{1/2} dE =: g_\phi(V_0(r)), \\ p_0(r) &= \frac{2^{7/2}}{3} \pi \int_{V_0(r)}^{\infty} \phi(E) (E - V_0(r))^{3/2} dE =: h_\phi(V_0(r)). \end{aligned}$$

Hence the ansatz (4.2) reduces the stationary Vlasov-Poisson system to the semilinear Poisson equation

$$\Delta V_0 = 4\pi g_\phi(V_0). \quad (4.3)$$

A solution of the latter automatically induces a solution of the static Euler-Poisson system which equation of state  $p_0 = h_\phi \circ g_\phi^{-1}(\rho_0)$ ; both  $g_\phi$  and  $h_\phi$  are invertible on their support under mild assumptions on  $\phi$ , cf. [24]. On the other hand, starting from the static, spherically symmetric Euler-Poisson system one can integrate the Euler equation using the equation of state and express  $\rho_0$  as a function of the potential  $V_0$  so that again the problem is reduced to a semilinear Poisson equation, which is precisely (4.3) if the equation of state has the form  $p_0 = h_\phi \circ g_\phi^{-1}(\rho_0)$ .

However, these parallels between static solutions of the two systems break down at several points: First of all, static solutions of the Euler-Poisson system, that is, steady states with vanishing velocity field, must be spherically symmetric, cf. [14], but for the Vlasov-Poisson system there exist axially symmetric steady states which are not spherically symmetric and which have non-vanishing velocity field  $\int v f_0 dv \neq 0$ , cf. [21].

Another point where the two systems differ, now concerning the question of stability, is the following: For the Euler-Poisson system the threshold  $n < 3$  which corresponds to  $\gamma > 4/3$  for the equation of state is sharp, since for

the polytropic case with  $\gamma=4/3$  the energy of a steady state is zero and an arbitrarily small perturbation of such a state can make the energy positive and cause part of the system to travel off to infinity. These assertions are shown in [3, Thm. 1.3 (iv), Thm. 1.4]. For the Vlasov-Poisson system one can go beyond this threshold to obtain stability provided  $0 < k < 7/2$  which corresponds to  $3/2 < n = k + 3/2 < 5$ . This was done in [10] by minimizing the energy  $E_{\text{kin}} + E_{\text{pot}}$  under the mass-Casimir constraint  $\int f + \mathcal{C}(f) = M$ . Moving the Casimir functional  $\mathcal{C}$  from the functional which we minimize into the constraint allows for an extension of the stability result which includes all the polytropes with index up to and including  $n = k + 3/2 = 5$ . The reduction mechanism which takes us to the Euler-Poisson system does no longer work in this context,  $\int \Phi(\rho) dx$  is not conserved by itself so that we cannot move this functional from the minimized functional into the constraint, and all this fits with the fact that—as opposed to the Vlasov-Poisson system—the threshold  $n < 3$  is sharp for the Euler-Poisson system as regards stability.

To conclude we note that if the variational formulation is chosen as in [10] then one can prove the stability result in the form stated in Theorem 3 without assuming uniqueness or even isolatedness of the minimizer, cf. [26]. Whether this is possible also in the framework of the present note is one of the many open problems in this area.

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